Inventory Model for Exponentially Increasing Demand with Time and Deterioration and Permissible Delay in Payments

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ABSTRACT

In this paper, an inventory model is developed in which demand is exponentially increasing with time and deterioration is taken non instantaneous. Realistic situation of permissible delay is also taken into consideration. Three different cases have been discussed for different situations. Expressions are obtained for total optimal cost in different situations. Three different algorithms are given to obtain the optimal solution. Cost minimization technique is used to solve the model.

Keywords - **Inventory model, Non instantaneous deterioration shortages, permissible delay in payments Finance,**

I. INTRODUCTION

In today's business transaction it is more and more common to see that the purchases are allowed a fixed time period before they settle the account with the supplier. This provides an advantage to the purchaser, due to the fact that they do not have to pay the supplier immediately after receiving the items but instead can defer their payment until the end of the allowed period. Thus paying later indirectly reduces the purchases cost of the items. On the other hand the permissible delay in payments produces benefit to the supplier such as it should attract new purchasers who consider it to be a type of price reduction. It is important to control and maintain the inventories of deteriorating items for the modern corporation. In general deterioration is defined as the damage, spoilage, dryness, vaporization etc. that results in decreases of usefulness of the original one. Inventory problems for deteriorating items have been widely studied by **Ghare** and **Schrader (1963).** They presented an EOQ model for an exponentially decaying inventory. Later **Covert** and **Philip (1973)** formulated the model with variable deterioration rate with two parameter **Weibull** distribution. **Philip (1974)** developed an inventory model with three parameter **Weibull** distribution rate and no shortages. **Shah (1977)** extends **Philip (1974)** model and considered that shortages are allowed. Recently **Goyal** and **Giri (2001)** provides a detailed review of deteriorating inventory literature. In the mentioned literature almost all the inventory models for deteriorating items assume that the deterioration occurs as soon as the retailer receives the commodities. However in real life most of goods would have a span of maintaining quality of the original condition, namely, during that period, there was no deterioration occurring. Such deterioration is treated as non instantaneous deterioration.

In the present model, an inventory model is developed in which demand is exponentially increasing with time and deterioration is taken non instantaneous. Realistic situation of permissible delay is also taken into consideration. Three different cases have been discussed for different situations. Expressions are obtained for total optimal cost in different situations. Three different algorithms are given to obtain the optimal solution. Cost minimization technique is used to solve the model.

II. NOTATIONS

1. C: Ordering cost of inventory per order.

2. C_1: Holding cost excluding interest charge per unit per unit time.

3. C_2: Shortage cost per unit per unit time.

4. C_3: Unit purchase cost.

5. Ir: Interest paid per rupee invested in stock s per year Ir>Ie.

6. Ie: Interest which can be earned per rupee per year.

7. q (t) : Inventory level at time t.

8. M : Permissible Delay period for settling accounts in time 0<M<1.

9. t_1: Time at which shortages starts.

10. T: Length of replenishment cycle.

11. μ : Life period of item at the end of which deterioration starts.

12. Q: Total amount of inventory produced or purchased at the beginning of each production cycle.

13. S(S<Q): Initial amount of inventory after fulfilling back orders.

14. TC (t_1, T): The total average cost of the inventory system per unit time.

15. $TC_{1\,(a)}$ (t_1, T): The total average cost of the inventory system per unit time for $M \le t_1$ and $M \le \mu$.

16. $TC_{1\,(b)}$ (t_1, T): The total average cost of the inventory system for $M \le t_1$ and $M \ge \mu$.

17. TC_2 (t_1, T): The total average cost of the inventory system per unit time for $M>t_1$.

III. ASSUMPTIONS

1. The inventory system consists of single item only.

2. There is no repair or replacement of the deteriorated unit.

3. The replenishment occurs instantaneously at an infinite rate.

4. When produced or purchased items arrive in stock they are fresh and new. They begin to deteriorate after a fixed time interval μ .

The deterioration function θ(t) is taken in the following form

$$\theta(t) = \alpha\beta t^{\beta-1}\, H(t-\mu) \quad (0<\alpha<<1)$$

$$\beta \ge 1$$

$$t, \mu > 0$$

where H (t - μ) is heaviside function defined as

$$H(t-\mu)\begin{cases} = 1 & t \ge \mu \\ = 0 & t < \mu \end{cases}$$

5. Demand rate (D(t)) is known and increases exponentially at time t, $t \ge 0$

$D(t) = ae^{bt} \quad a \ge 0$

where a is the initial demand and b is a constant governing the increasing rate of demand.

6. Shortages are allowed and only a fraction λ ($0 < \lambda < 1$) of the demand during the stock out period is backlogged and the remaining fraction ($1 - \lambda$) is lost.

7. During the fixed credit period M the unit cost of generated sales revenue is deposited in an interest bearing account. The difference between sales price and unit cost is retained by the system to meet the day to day expenses of the system. At the end of the credit period account is to be settled. Then interest is again earned during the period (M, t_1). If $M \le t$, interest charges are paid on the stock held beyond the permissible period.

IV. MATHEMATICAL EQUATIONS

$$\frac{dq(t)}{dt} = -a e^{bt} \qquad ,0 \le t \le \mu \tag{1}$$

$$\frac{dq(t)}{dt} + \theta_0\, q(t) = -a e^{bt} \qquad ,\mu \le t \le t_1 \tag{2}$$

$$\frac{dq(t)}{dt} = -a e^{bt} \qquad , t_1 \le t \le T. \qquad (3)$$

Boundary conditions are at $t = 0, \quad q(t) = S$

at $t = \mu, \quad q(t) = q(\mu)$

at $t = t_1, \quad q(t_1) = 0$.

Solution of equation (1) is given by

$$q(t) = -a \int e^{bt} \, dt + A$$

where A is the constant of integration.

or $$q(t) = -\frac{a}{b} e^{bt} + A.$$

Using boundary condition at $t = 0, q(t) = S$,

we get

$$S = -\frac{a}{b} + A.$$

Therefore

$$q(t) = S + \frac{a}{b}\left(1 - e^{bt}\right) \qquad , 0 \le t \le \mu \qquad (4)$$

Also at $t = \mu$ equation (4) reduces to

$$q(\mu) = S + \frac{a}{b}\left(1 - e^{b\mu}\right) \qquad (5)$$

Solution of equation (2) is given by

$$q(t) e^{\theta_0 t} = -a \int e^{(b+\theta_0)t} \, dt + B$$

where B is constant of integration.

Using boundary condition at $t = \mu \quad , q(t) = q(\mu)$, one can get

$$q(t) e^{\theta_0 t} = \frac{a}{(b+\theta_0)}\left[e^{(b+\theta_0)\mu} - e^{(b+\theta_0)t}\right] + \left[S + \frac{a}{b}(1 - b\mu)\right] e^{\theta_0 \mu} \qquad (6)$$

Using boundary condition at t=t1, Q(t1)=0 from equation (6) the value of S is given by

$$S = \frac{a}{(b+\theta_0)}\left[e^{(b+\theta_0)\mu} - e^{(b+\theta_0)t}\right] e^{-\theta_0 \mu} - \frac{a}{b}(1 - b\mu) \qquad (7)$$

Now substituting the value of S from equation (7) in equation (6), one can get

$$q(t) = \frac{a}{(b+\theta_0)}\left[e^{bt_1 + \theta_0(t_1 - t)} - e^{bt}\right] \qquad \mu \le t \le t_1 \qquad (8)$$

Solution of equation (3) is given by

$$q(t) = \frac{-a}{b} e^{bt} + E$$

where E is a constant of integration. Applying the boundary condition at $t = t_1 \quad , q(t) = 0$, one can get

$$q(t) = \frac{a}{b}\left[e^{bt_1} - e^{bt}\right] \qquad t_1 \le t \le T \qquad (9)$$

Total amount of holding units (q_H) during the period $(0, t_1)$ is

$$q_H = \int_0^{\mu} q(t)\,dt + \int_{\mu}^{t_1} q(t)\,dt$$

$$= \int_0^{\mu} \left\{S + \frac{a}{b}\left(1 - e^{bt}\right)\right\} dt + \int_{\mu}^{t_1} \frac{a e^{-\theta_0 t}}{(b+\theta_0)}\left[e^{(b+\theta_0)t_1} - e^{(b+\theta_0)t}\right] dt$$

$$= \left[S\mu + \frac{a}{b}\mu - \frac{a}{b^2}\left(e^{b\mu} - 1\right)\right] + \frac{a}{(b+\theta_0)}$$

$$\left[-\frac{e^{bt_1}}{\theta_0} + \frac{e^{bt_1}}{b} + \frac{e^{b+\theta_0(t_1-\mu)}}{\theta_0} - \frac{e^{b\mu}}{b}\right]$$

Substituting the value of S from (7) , we have

$$q_H = \left[\frac{a\mu}{(b+\theta_0)}\left[e^{(b+\theta_0)t_1} - e^{(b+\theta_0)\mu}\right] e^{-\theta_0 \mu} - \frac{a\mu}{b}(1 - b\mu) + \frac{a}{b}\mu\right.$$

$$-\frac{a}{b^2}\left(e^{b\mu}-1\right)\Bigg]+\frac{a}{(b+\theta_0)}\left[\frac{e^{bt_1}}{-\theta_0}+\frac{e^{bt_1}}{b}+\frac{e^{b+\theta_0(t_1-\mu)}}{\theta_0}-\frac{e^{b\mu}}{b}\right]$$

$$=\frac{a\mu}{(b+\theta_0)}\left[e^{b+\theta_0(t_1-\mu)}-e^{b\mu}\right]+a\mu^2-\frac{a}{b^2}\left(e^{b\mu}-1\right)$$

$$+\frac{a}{(b+\theta_0)}\left[\frac{e^{bt_1}}{-\theta_0}+\frac{e^{bt_1}}{b}+\frac{e^{b+\theta_0(t_1-\mu)}}{\theta_0}-\frac{e^{b\mu}}{b}\right] \quad (10)$$

Total amount of deteriorated units (q_D) during the period $(0, t_1)$ is

$$q_D = q(\mu) - \int_{\mu}^{t_1} ae^{bt}\, dt$$

$$= S + \frac{a}{b}\left(1-e^{b\mu}\right) - \frac{a}{b}\left[e^{bt}\right]_{\mu}^{t_1}$$

$$=\frac{a}{(b+\theta_0)}\left[e^{(b+\theta_0)t_1}-e^{(b+\theta_0)\mu}\right]e^{-\theta_0\mu}-\frac{a}{b}\left[e^{bt_1}-e^{b\mu}\right] \quad (11)$$

Amount of shortage units (q_S) during the period (t_1, T) is given by

$$q_S = -\int_{t_1}^{T} q(t)\, dt$$

$$= -\int_{t_1}^{T} \frac{a}{b}\left(e^{bt_1}-e^{bt}\right)dt$$

$$= -\left[\frac{a}{b}\left(e^{bt_1}t-\frac{e^{bt}}{b}\right)\right]_{t_1}^{T}$$

$$=-\left[\frac{a}{b}e^{bt_1}(T-t_1)-\frac{1}{b}\left(e^{bT}-e^{bt_1}\right)\right] \quad (12)$$

Now there are two possibilities regarding the period M of permissible delay in payments.

CASE I : $M \le t_1$

CASE II : $M > t_1$

CASE I : $M \le t_1$

The case (1) is further divided into two sub cases i.e. case I(a) and case I(b)

CASE I(a) : $M \le \mu \le t_1$

CASE I(b) : $\mu \le M \le t_1$

CASE I(a) :

Since here the length of period with positive inventory stock is larger than the credit period M, the buyer can use sale revenue to earn the interest with an annual rate Ie during the period [0,M]. The unit cost of the generated sales revenue is deposited in an interest bearing account. The difference between sales price and unit cost is retained by the system to meet the day to day expenses of the system. At the end of the credit period, the account is settled. After setting the account at time M again the unit cost of generated sales revenue is deposited in an interest bearing account to earn interest with an annual rate Ie during the period [M,t_1]. Beyond the fixed credit period product still in stock is assumed to be financed with an annual rate Ir. Now the total interest earned $IE_{1(a)}$ during the period $[0, t_1]$ is given by

$$IE_{1(a)} = C_3 I_e\left[\int_0^M (M-t)ae^{bt}dt+\int_M^{t_1}(t_1-t)ae^{bt}dt\right]$$

$$= C_3 I_e\left[\left\{\frac{ae^{bt}}{b}\left(M-t+\frac{1}{b}\right)\right\}_0^M+\frac{ae^{bt}}{b}\left(t_1-t+\frac{1}{b}\right)_M^{t_1}\right]$$

$$=C_3I_e\left[\frac{ae^{bM}}{b^2}-\frac{a}{b}\left(M+\frac{1}{b}\right)+\frac{ae^{bt_1}}{b^2}-\frac{ae^{bM}}{b}\left(t_1-M+\frac{1}{b}\right)\right]$$

$$= C_3 I_e\left[\frac{ae^{bt_1}}{b^2}-\frac{a}{b}\left(M+\frac{1}{b}\right)-\frac{ae^{bM}}{b}(t_1-M)\right] \quad (13)$$

Total interest payable $IP_{1(a)}$ is given by

$$IP_{1(a)} = C_3 I_r \int_M^{t_1} q(t)\, dt$$

$$= C_3 I_r\left[\int_M^{\mu} q(t)\, dt + \int_{\mu}^{t_1} q(t)\, dt\right]$$

$$= C_3 I_r\left[\int_M^{\mu}\left\{S+\frac{a}{b}(1-e^{bt})\right\}dt\right.$$

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$$+\int_{\mu}^{t_1}\frac{a}{(b+\theta_0)}\Big\{e^{bt_1+\theta_0(t_1-t)}-e^{bt}\Big\}dt\Big]$$

$$=C_3I_r\Bigg[\Big\{St+\frac{a}{b}\Big(t-\frac{e^{bt}}{b}\Big)\Big\}_M^{\mu}+\frac{a}{(b+\theta_0)}\Big\{\frac{e^{bt_1+\theta_0(t_1-t)}}{-\theta_0}-\frac{e^{bt}}{b}\Big\}_{\mu}^{t_1}\Bigg]$$

$$=C_3I_r\Big[S(\mu-M)+\frac{a}{b}(\mu-M)-\frac{a}{b^2}\big(e^{b\mu}-e^{bM}\big)+\frac{a}{(b+\theta_0)}$$

$$\Big\{\frac{1}{-\theta_0}\Big(e^{bt_1}-e^{bt_1+\theta_0(t_1-\mu)}\Big)-\frac{1}{b}\Big(e^{bt_1}-e^{b\mu}\Big)\Big\}\Big]$$

$$=C_3I_r\Big[\frac{a(\mu-M)}{(b+\theta_0)}\Big\{e^{b+\theta_0(t_1-\mu)}-e^{b\mu}\Big\}+a(\mu-M)\mu$$

$$-\frac{a}{b^2}\big(e^{b\mu}-e^{bM}\big)+\frac{a}{(b+\theta_0)}$$

$$\Big\{\frac{1}{-\theta_0}\Big(e^{bt_1}-e^{bt_1+\theta_0(t_1-\mu)}\Big)-\frac{1}{b}\Big(e^{bt_1}-e^{b\mu}\Big)\Big\}\Big] \qquad (14)$$

Therefore total average cost in this case is

$$TC_{1(a)}(t_1,T)=\frac{C+C_1q_H+C_3q_D+C_2q_S+IP_{1(a)}-IE_{1(a)}}{T}$$

$$=\frac{C}{T}+\frac{C_1}{T}\Big[\frac{a\mu}{(b+\theta_0)}\Big\{e^{b+\theta_0(t_1-\mu)}-e^{b\mu}\Big\}$$

$$+a\mu^2-\frac{a}{b^2}\big(e^{b\mu}-1\big)+\frac{a}{(b+\theta_0)}\Big\{\frac{e^{bt_1}}{-\theta_0}+\frac{e^{bt_1}}{b}$$

$$\frac{e^{b+\theta_0(t_1-\mu)}}{\theta_0}-\frac{e^{b\mu}}{b}\Big\}\Big]+\frac{C_3}{T}\Big[\frac{a}{(b+\theta_0)}\Big\{e^{b+\theta_0(t_1-\mu)}-e^{b\mu}\Big\}$$

$$\frac{1}{b}\Big(e^{bT}-e^{bt_1}\Big)\Big]+\frac{C_3I_r}{T}\Big[\frac{a(\mu-M)}{(b+\theta_0)}\Big\{e^{b+\theta_0(t_1-\mu)}-e^{b\mu}\Big\}$$

$$+a\mu(\mu-M)-\frac{a}{b^2}\big(e^{b\mu}-e^{bM}\big)-\frac{a}{(b+\theta_0)}$$

$$\Big\{\frac{1}{\theta_0}\Big(e^{bt_1}-e^{bt_1+\theta_0(t_1-\mu)}\Big)-\frac{1}{b}\Big(e^{bt_1}-e^{b\mu}\Big)\Big\}\Big]$$

$$-\frac{C_3I_e}{T}\Big[\frac{ae^{bt_1}}{b^2}-\frac{a}{b}\Big(M+\frac{1}{b}\Big)-\frac{ae^{bM}}{b}(t_1-M)\Big] \qquad (15)$$

To minimize the total average cost per unit time $TC_{1(a)}$ (t_1, T) the optimal values of t_1 and T (say t_1* and T*) can be obtained by solving the following two equations simultaneously

$$\frac{\partial\, TC_{1(a)}(t_1,T)}{\partial\, t_1}=0 \qquad (16)$$

and

$$\frac{\partial\, TC_{1(a)}(t_1,T)}{\partial\, T}=0 \qquad (17)$$

provided they satisfy the sufficient conditions

$$\frac{\partial^2\, TC_{1(a)}(t_1,T)}{\partial\, t_1^2}>0$$

$$\frac{\partial^2\, TC_{1(a)}(t_1,T)}{\partial\, T^2}>0$$

and

$$\left(\frac{\partial^2 TC_{1(a)}(t_1,T)}{\partial t_1^2}\right)\left(\frac{\partial^2 TC_{1(a)}(t_1,T)}{\partial T^2}\right)-\left(\frac{\partial^2 TC_{1(a)}(t_1,T)}{\partial t\,\partial T}\right)^2>0$$

Equations (16) and (17) are equivalent to

$$\frac{C_1}{T}\Big[\frac{a\mu}{(b+\theta_0)}\Big\{\theta_0\, e^{b+\theta_0(t_1-\mu)}\Big\}+\frac{a}{(b+\theta_0)}\Big\{\frac{b\,e^{bt_1}}{-\theta_0}+e^{bt_1}+$$

$$-\frac{a}{b}\Big(e^{bt_1}-e^{b\mu}\Big)\Big]+\frac{C_2}{T}\Big[\frac{a}{b}e^{bt_1}(t_1-T)+$$

$$\frac{e^{b+\theta_0(t_1-\mu)}}{\theta_0^2}\Big\}\Big]+\frac{C_3}{T}\Big[\frac{a}{(b+\theta_0)}\theta_0\, e^{b+\theta_0(t_1-\mu)}-ae^{bt_1}\Big]$$

$$\frac{C_3}{T}\Big[\frac{a}{(b+\theta_0)}\theta_0\, e^{b+\theta_0(t_1-\mu)}-ae^{bt_1}\Big]+\frac{C_2}{T}\Big[ae^{bt_1}(-t_1+T)\Big]$$

$$\frac{C_3 I_r}{T}\left[\frac{a(\mu - M)}{(b+\theta_0)} e^{b+\theta_0(t_1-\mu)}\theta_0 - \frac{a}{(b+\theta_0)}\right.$$

$$\left.\left\{\frac{b}{\theta_0}e^{bt_1} - \frac{(b+\theta_0)}{\theta_0}e^{bt_1+\theta_0(t_1-\mu)} - e^{bt_1}\right\}\right]$$

$$-\frac{C_3 I_e}{T}\left[\frac{a\ e^{bt_1}}{b} - \frac{a\ e^{bM}}{b}\right] \quad (18)$$

and

$$-\frac{1}{T^2}\left[C + C_1\left[\frac{a\mu}{(b+\theta_0)}\left\{e^{b+\theta_0(t_1-\mu)} - e^{b\mu}\right\} + a\mu^2\right.\right.$$

$$\left.-\frac{a}{b^2}\left(e^{b\mu}-1\right) + \frac{a}{(b+\theta_0)}\left\{\frac{e^{bt_1}}{-\theta_0} + \frac{e^{bt_1}}{b}\frac{e^{b+\theta_0(t_1-\mu)}}{\theta_0} - \frac{e^{b\mu}}{b}\right\}\right]$$

$$+C_3\left[\frac{a}{(b+\theta_0)}\left\{e^{b+\theta_0(t_1-\mu)} - e^{b\mu}\right\} - \frac{a}{b}\left(e^{bt_1} - e^{b\mu}\right)\right]$$

$$+C_2\left[\frac{a}{b}e^{bt_1}(t_1 - T) + \frac{1}{b}\left(e^{bT} - e^{bt_1}\right)\right] + C_3 I_r\left[\frac{a(\mu - M)}{(b+\theta_0)}\right.$$

$$\left\{e^{b+\theta_0(t_1-\mu)} - e^{b\mu}\right\} + a\mu(\mu - M) - \frac{a}{b^2}\left(e^{b\mu} - e^{bM}\right) - \frac{a}{(b+\theta_0)}$$

$$\left.\left.\left\{\frac{1}{\theta_0}\left(e^{bt_1} - e^{bt_1+\theta_0(t_1-\mu)}\right) - \frac{1}{b}\left(e^{bt_1} - e^{b\mu}\right)\right\}\right] - C_3 I_e\left[\frac{ae^{bt_1}}{b^2}\right.$$

$$\left.\left.-\frac{a}{b}\left(M + \frac{1}{b}\right) - \frac{ae^{bM}}{b}(t_1 - M)\right] + \frac{C_2}{T}\left[\frac{-a}{b}e^{bt_1} + \frac{a}{b}e^{bM}\right]\right]$$

$$(19)$$

To get the optimal value of t_1 and T which minimizes total cost $TC_{1(a)}(t_1, T)$ one need to develop the following algorithm to find the optimal (t_1, T)

ALGORITHM 1(p):

STEP 1: Perform (I) to (IV)

(I) Start with $t_1 = M$

(II) Substitute $t_{1(1)}$ in equation (18) to obtain $T_{(1)}$

(III) Using $T_{(1)}$ determines $t_{1(2)}$ from equation (19)

(IV) Repeat (II) and (III) until no change occurs in the value of t_1 and T.

STEP 2: To compare t_1 and M

(I) If $M \le t_1$, t_1 is feasible than go to step (3).

(II) If $M \le t_1$, t_1 is not feasible set $t_1 = M$ and evaluate the corresponding values of T from equation (19) and then go to the step (3).

STEP 3: Calculate the corresponding total cost.

$TC_{1(a)}(t_1^*, T^*)$

CASE I(b): $M \ge \mu$ and $M \le t_1$

This case is similar to case I(a). But as $M > \mu$ the interest earned $IE_{1(b)}$ during $[0, t_1]$ is given by

$$IE_{1(b)} = C_3 I_e\left[\int_0^M (M - t)ae^{bt} + \int_M^{t_1}(t_1 - t)ae^{bt}dt\right]$$

$$= C_3 I_e\left[\frac{ae^{bt_1}}{b^2} - \frac{a}{b}\left(M + \frac{1}{b}\right) - \frac{ae^{bM}}{b}(t_1 - M)\right] \quad (20)$$

Interest payable $IP_{1(b)}$ for the period $[M, t_1]$ is given by

$$IP_{1(b)} = C_3 I_r \int_M^{t_1} q(t)\, dt$$

$$= C_3 I_r \int_M^{t_1} \frac{a}{(b+\theta_0)}\left[e^{bt_1+\theta_0(t_1-t)} - e^{bt}\right]dt$$

$$= C_3 I_r\left[\frac{a}{(b+\theta_0)}\left\{\frac{e^{bt_1+\theta_0(t_1-t)}}{-\theta_0} - \frac{e^{bt}}{b}\right\}_M^{t_1}\right]$$

$$= C_3 I_r\left[\frac{a}{(b+\theta_0)}\left\{\frac{e^{bt_1}}{-\theta_0} - \frac{e^{bt_1+\theta_0(t_1-M)}}{-\theta_0} - \frac{e^{bt_1}}{b} + \frac{e^{bM}}{b}\right\}\right] \quad (21)$$

Now the total average cost $TC_{1(b)}(t_1, T)$ in this case is given by

$$TC_{1(b)}(t_1, T) = \frac{C + C_1 q_H + C_3 q_D + C_2 q_S + IP_{1(b)} - IE_{1(b)}}{T}$$

$$= \frac{C}{T} + \frac{C_1}{T}\left[\frac{a\mu}{(b+\theta_0)}\left\{e^{b+\theta_0(t_1-\mu)} - e^{b\mu}\right\} + a\mu^2\right.$$

$$-\frac{a}{b^2}\left(e^{b\mu}-1\right) + \frac{a}{(b+\theta_0)}\left\{\frac{e^{bt_1}}{-\theta_0} + \frac{e^{bt_1}}{b}\frac{e^{b+\theta_0(t_1-\mu)}}{\theta_0}\right.$$

$$\left.\left.-\frac{e^{b\mu}}{b}\right\}\right] + \frac{C_3}{T}\left[\frac{a}{(b+\theta_0)}\left\{e^{b+\theta_0(t_1-\mu)} - e^{b\mu}\right\}\right.$$

$$\left.-\frac{a}{b}\left(e^{bt_1}-e^{b\mu}\right)\right] + \frac{C_2}{T}\left[\frac{a}{b}e^{bt_1}(t_1-T) + \frac{1}{b}\left(e^{bT}-e^{bt_1}\right)\right]$$

$$+\frac{C_3 I_r}{T}\left[\frac{a}{(b+\theta_0)}\left\{\frac{e^{bt_1}}{-\theta_0} + \frac{e^{bt_1+\theta_0(t_1-M)}}{\theta_0} - \frac{e^{bt_1}}{b} + \frac{e^{bM}}{b}\right\}\right]$$

$$-\frac{C_3 I_e}{T}\left[\frac{ae^{bt_1}}{b^2} - \frac{a}{b}\left(M+\frac{1}{b}\right) - \frac{ae^{bM}}{b}(t_1-M)\right] \quad (22)$$

To minimize the total average cost per unit time $TC_{1(b)}(t_1, T)$ the optimal values of t_1 and T (say t_1* and T*) can be obtained by solving the following two equations simultaneously

$$\frac{\partial\, TC_{1(a)}(t_1, T)}{\partial\, t_1} = 0 \quad (23)$$

and
$$\frac{\partial\, TC_{1(a)}(t_1, T)}{\partial\, T} = 0 \quad (24)$$

provided they satisfy the sufficient conditions

$$\frac{\partial^2\, TC_{1(a)}(t_1, T)}{\partial\, t_1^2} > 0 ,$$

$$\frac{\partial^2\, TC_{1(a)}(t_1, T)}{\partial\, T^2} > 0$$

and

$$\left(\frac{\partial^2 TC_{1(a)}(t_1,T)}{\partial t_1^2}\right)\left(\frac{\partial^2 TC_{1(a)}(t_1,T)}{\partial T^2}\right) - \left(\frac{\partial^2 TC_{1(a)}(t_1,T)}{\partial t\, \partial T}\right)^2 > 0 \cdot$$

Equation (23) and (24) are equivalent to

$$\frac{C_1}{T}\left[\frac{a\mu}{(b+\theta_0)}\theta_0 e^{(b+\theta_0)(t_1-\mu)} + \frac{a}{(b+\theta_0)}\left\{\frac{b\,e^{bt_1}}{-\theta_0} + e^{bt_1} + \right.\right.$$

$$\left.\left.\frac{e^{b+\theta_0(t_1-\mu)}}{\theta_0^{\,2}}\right\}\right] + \frac{C_3}{T}\left[\frac{a}{(b+\theta_0)}\theta_0\, e^{b+\theta_0(t_1-\mu)} - ae^{bt_1}\right]$$

$$+\frac{C_2}{T}\left[a\, e^{bt_1}(T-t_1)\right] + \frac{C_3 I_r}{T}\left[\frac{a}{(b+\theta_0)}\left\{\frac{be^{bt_1}}{-\theta_0}\right.\right.$$

$$\left.\left.-e^{t_1(b+\theta_0)-\theta_0 M}(b+\theta_0) - e^{bt_1}\right\}\right] - \frac{C_3 I_e\, a}{T\,b}\left[a\, e^{bt_1} - e^{bM}\right] = 0 \quad (25)$$

and

$$-\frac{1}{T^2}\left[C + C_1\left[\frac{a\mu}{(b+\theta_0)}\left\{e^{b+\theta_0(t_1-\mu)} - e^{b\mu}\right\} + a\mu^2\right.\right.$$
$$\left.-\frac{a}{b^2}\left(e^{b\mu}-1\right) + \frac{a}{(b+\theta_0)}\left\{\frac{e^{bt_1}}{-\theta_0} + \frac{e^{bt_1}}{b}\frac{e^{b+\theta_0(t_1-\mu)}}{\theta_0} - \frac{e^{b\mu}}{b}\right\}\right]$$

$$+C_3\left[\frac{a}{(b+\theta_0)}\left\{e^{b+\theta_0(t_1-\mu)} - e^{b\mu}\right\} - \frac{a}{b}\left(e^{bt_1}-e^{b\mu}\right)\right]$$

$$+C_2\left[\frac{a}{b}e^{bt_1}(t_1-T) + \frac{1}{b}\left(e^{bT}-e^{bt_1}\right)\right] + C_3 I_r\left[\frac{a(\mu-M)}{(b+\theta_0)}\right.$$

$$\left\{e^{b+\theta_0(t_1-\mu)} - e^{b\mu}\right\} + a\mu(\mu-M) - \frac{a}{b^2}\left(e^{b\mu}-e^{bM}\right) - \frac{a}{(b+\theta_0)}$$

$$\left.\left.\left\{\frac{1}{\theta_0}\left(e^{bt_1}-e^{bt_1+\theta_0(t_1-\mu)}\right) - \frac{1}{b}\left(e^{bt_1}-e^{b\mu}\right)\right\}\right] - C_3 I_e\left[\frac{ae^{bt_1}}{b^2}\right.$$

$$\left.-\frac{a}{b}\left(M+\frac{1}{b}\right) - \frac{ae^{bM}}{b}(t_1-M)\right] + \frac{C_2}{T}\left[\frac{-a}{b}\left(e^{bt_1}-e^{bT}\right)\right] = 0 \quad (26)$$

Now we develop the algorithm to find the optimal values of t_1 and T.

ALGORITHM 1(q):

STEP1: PERFORM (I) TO (IV)

(I) Start with $t_{1(1)} = M$

(II) Substitute $t_{1(1)}$ into equation (25) to evaluate $T_{(1)}$

(III) Using $T_{(1)}$ to determine $t_{1(2)}$ from equation (26)

(IV) Repeat (II) and (III) until no change occurs in the value of t_1 and T.

STEP2: To compare t_1 and M

(I) If $M \le t_1$, then t_1 is feasible than go to step (3).

(II) If $M > t_1$, then t_1 is not feasible. Set $t_1 = M$ and evaluate the corresponding values of T from equation (26) and then go to the step (3).

STEP 3: Compute the corresponding $TC_{1(b)}(t_1^*, T^*)$.

CASE (2): $t_1 < M$

In this case since $t_1 < M$ the buyer pays no interest and earns the interest during the period [0, M], The interest earned in this case is denoted by $IE_{(2)}$ and is

$$IE_{(2)} = C_3 I_e \int_0^{t_1} (M - t) a e^{bt} dt$$

$$= C_3 I_e \left[\frac{Mae^{bt}}{b} - \frac{tae^{bt}}{b} + \frac{e^{bt}}{b^2} \right]_0^{t_1}$$

$$= C_3 I_e \left[\frac{Ma}{b}\left(e^{bt} - 1\right) - \frac{a}{b} t_1 e^{bt_1} + \frac{1}{b^2}\left(e^{bt_1} - 1\right) \right] \quad (27)$$

The total average cost per unit time $TC_2(t_1, T)$ in this case is

$$TC_2(t_1, T) = \frac{C + C_1 q_H + C_3 q_D + C_2 q_S - IE_2}{T}$$

$$= \frac{C}{T} + \frac{C_1}{T}\left[\frac{a\mu}{(b+\theta_0)}\left\{ e^{b+\theta_0(t_1-\mu)} - e^{b\mu} \right\} + a\mu^2 - \frac{a}{b^2}\left(e^{b\mu} - 1\right) + \frac{a}{(b+\theta_0)}\left\{ \frac{e^{bt_1}}{-\theta_0} + \frac{e^{bt_1}}{b} \frac{e^{b+\theta_0(t_1-\mu)}}{\theta_0} - \frac{e^{b\mu}}{b} \right\} \right] + \frac{C_3}{T}\left[\frac{a}{(b+\theta_0)}\left\{ e^{b+\theta_0(t_1-\mu)} - e^{b\mu} \right\} - \frac{a}{b}\left(e^{bt_1} - e^{b\mu}\right) \right] + \frac{C_2}{T}\left[\frac{a}{b} e^{bt_1}(t_1 - T) + \frac{1}{b}\left(e^{bT} - e^{bt_1}\right) \right]$$

$$- \frac{C_3 I_e}{T}\left[\frac{Ma}{b}\left(e^{bt_1} - 1\right) - \frac{a}{b} t_1 e^{bt_1} + \frac{1}{b^2}\left(e^{bt_1} - 1\right) \right] \quad (28)$$

To minimize the total average cost per unit time $TC_{1(a)}(t_1, T)$ the optimal values of t_1 and T (say t_1^* and T*) can be obtained by solving the following two equations simultaneously

$$\frac{\partial TC_2(t_1, T)}{\partial t_1} = 0 \quad (29)$$

and

$$\frac{\partial TC_2(t_1, T)}{\partial T} = 0 \quad (30)$$

provided they satisfy the sufficient conditions

$$\frac{\partial^2 TC_2(t_1, T)}{\partial t_1^2} > 0$$

$$\frac{\partial^2 TC_2(t_1, T)}{\partial T^2} > 0$$

and

$$\left(\frac{\partial^2 TC_2(t_1, T)}{\partial t_1^2}\right)\left(\frac{\partial^2 TC_2(t_1, T)}{\partial T^2}\right) - \left(\frac{\partial^2 TC_2(t_1, T)}{\partial t\, \partial T}\right)^2 \rangle 0$$

Equations (29) and (30) are equivalent to

$$\frac{C}{T} + \frac{C_1}{T}\left[\frac{a\mu}{(b+\theta_0)} \theta_0 e^{(b+\theta_0)(t_1-\mu)} + \frac{a}{(b+\theta_0)}\left\{ \frac{b e^{bt_1}}{-\theta_0} + e^{bt_1} + \frac{e^{b+\theta_0(t_1-\mu)}}{\theta_0^2} \right\} \right] + \frac{C_3}{T}\left[\frac{a}{(b+\theta_0)} \theta_0 e^{b+\theta_0(t_1-\mu)} - ae^{bt_1} \right]$$

$$\frac{C_2}{T}\left[ae^{bt_1}(T - t_1)\right] - \frac{C_3 I_e}{T}\left[Mae^{bt_1} - \frac{a}{b}\left(e^{bt_1} - t_1 be^{bt_1}\right) + \frac{1}{b} e^{bt_1} \right] = 0 \quad (31)$$

and

$$-\frac{1}{T^2}\left[C + C_1\left[\frac{a\mu}{(b+\theta_0)}\left\{ e^{b+\theta_0(t_1-\mu)} - e^{b\mu} \right\} + a\mu^2 \right.\right.$$

$$-\frac{a}{b^2}\left(e^{b\mu}-1\right)+\frac{a}{(b+\theta_0)}\left\{\frac{e^{bt_1}}{-\theta_0}+\frac{e^{bt_1}}{b}\frac{e^{b+\theta_0(t_1-\mu)}}{\theta_0}-\frac{e^{b\mu}}{b}\right\}\Bigg]\Bigg]$$

$$+C_3\left[\frac{a}{(b+\theta_0)}\left\{e^{b+\theta_0(t_1-\mu)}-e^{b\mu}\right\}-\frac{a}{b}\left(e^{bt_1}-e^{b\mu}\right)\right]$$

$$+C_2\left[\frac{a}{b}e^{bt_1}(t_1-T)+\frac{1}{b}\left(e^{bT}-e^{bt_1}\right)\right]+C_3I_e\left[\frac{ae^{bt_1}}{b^2}\right.$$

$$\left.-\frac{a}{b}\left(M+\frac{1}{b}\right)-\frac{ae^{bM}}{b}(t_1-M)\right]+\frac{C_2}{T}\left[\frac{-a}{b}\left(e^{bt_1}-e^{bT}\right)\right]=0 \quad (32)$$

Now we develop the following algorithm to find the optimal values of t_1 and T.

ALGORITHM 2

STEP 1: PERFORM (I) TO (IV)

(I) Start with $(t_1)_1$ = M

(II) Substituting $t_{1(1)}$ = M into equation (31) to evaluate $T_{(1)}$

(III) Using $T_{(1)}$ to determine $t_{1(2)}$ from equation (32)

(IV) Repeat (II) and (III) until no change occurs in the value of t_1 and T.

STEP 2: Compare t_1 and M

(I) If t_1 < M, t_1 is feasible than go to step (3).

(II) If $t_1 \geq$ M, t_1 is not feasible set t_1 = M and evaluate the corresponding values of T from equation (32) and then go to the step (3).

STEP 3: As stated earlier, the objective of this problem is to determine the optimal values of t_1 and T so that TC (t_1, T) is minimum. As the discussion carried out so far one can get

$$TC\left(t_1^*,T^*\right)=Min\left\{TC_{1(a)}\left(t_1^*,T^*\right),TC_{1(b)}\left(t_1^*,T^*\right),TC_2\left(t_1^*,T^*\right)\right\}$$

V. EXAMPLE AND TABLES

NUMERICAL ILLUSTRATION :

Shortage cost C_2=3 Unit purchase cost C_3=12
Backlogging rate =0.75 Holding cost C_1=2.5
Interest earned I_e=0.06 Interest charged I_r=0.09
Ordering cost C=50
α = 10,000 β =2.5 α **>0,** β >1
T = 10, t_1 <=t<=T a =100
b = 0.5 $\gamma = 0.5$

For Case I: $t_1 < m$ **(** credit period**)**

Table 1

m	t_1	T	T C_1 (t_1,T)
4	2.42032	5.08439	553.069
5	2.45922	5.07385	566.889
6	2.49892	5.06536	580.782
7	2.53954	5.05911	594.713
8	2.58119	5.05527	608.640

For Case II: $t_1 \geq m$

Table 2

M	t_1	P	T C_2 (t_1,T)
4	8.43614	22.4571	34.3903
5	8.42040	21.7532	37.3821
6	8.40475	21.0053	41.0370
7	8.38919	20.2284	45.4291
8	8.37373	19.4369	50.6412

VI .OBSERVATION:

From Table 1, we can say that if permissible delay period is increase then the time of inventory period is also increase. But selling price is decrease and the profit is increase.

From Table 2, we can say that if permissible delay period is increase then the time of inventory period is also decrease. And selling price is decrease and the profit is increase.

VII. CONCLUSION

In this model, an appropriate pricing and lot sizing model for a retailer when the supplier provides a permissible delay in payments is developed and discussed. We desire the first and second order conditions for finding the optimal cost and then developed an algorithm to solve the problem. The model is proposed for non instantaneous deteriorating items with exponential demand. Shortages are allowed and they are partially backlogged. During the shortage period only a fraction of the demand is left. The algebraic procedure and cost minimization procedure is applied to find the different optimal values. In the end some particular cases are also given.

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